

Yangian symmetry of the $Y=0$ maximal giant graviton

Niall MacKay^{a *} and Vidas Regelskis^{a b †}

^a *Department of Mathematics, University of York,
Heslington, York YO10 5DD, UK*

^b *Vilnius University Institute of Theoretical Physics and Astronomy,
Goštauto 12, Vilnius 01108, Lithuania*

ABSTRACT: We study the remnants of Yangian symmetry of AdS/CFT magnons reflecting from boundaries with no degrees of freedom. We present the generalized twisted boundary Yangian of open strings ending on boundaries which preserve only a subalgebra \mathfrak{h} of the bulk algebra \mathfrak{g} , where $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair. This is realized by open strings ending on the $D3$ brane known as the $Y = 0$ maximal giant graviton in $AdS_5 \times S^5$. We also consider the Yangian symmetry of the boundary which preserves an $\mathfrak{su}(1|2)$ subalgebra only.

*Email: nm15@york.ac.uk

†Email: vr509@york.ac.uk

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1. Introduction

Since the discovery of integrable structures [1, 2, 3] in the AdS/CFT correspondence [4], much use has been made of them on both sides, $\mathcal{N} = 4$ super Yang-Mills gauge theory and the $AdS_5 \times S^5$ superstring. The residual symmetry in light-cone quantization, the centrally extended $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ superalgebra [5], has played a very important role in understanding both sides of the correspondence and the underlying integrability. An important implication of integrability is that particle momenta are conserved in scattering, and that every scattering process factorizes into a sequence of two-particle interactions. Thus all scattering information is encrypted in the two-body S -matrix [6, 7, 8].

The requirement that the fundamental S -matrix be invariant under the symmetry algebra fixes it uniquely up to an overall phase factor [5] (which must respect unitarity

and crossing symmetry). But in addition to the fundamental particles, the spectrum of the string sigma model contains an infinite tower of bound states [9, 10, 11] appearing as poles of the S -matrix and in the Bethe ansatz equations. The construction of S -matrices for the bound states is more complicated, as the symmetry algebra alone is no longer sufficient to determine the S -matrices uniquely. Further constraints are required, arising from either the Yang-Baxter equation or the underlying Yangian symmetry [12, 13].

Yangians are important algebraic structures which appear in many integrable models [14, 15], typically as a hidden extension of Lie symmetry. They are deformations of the polynomial algebra of the Lie algebra, and are originally associated with integrable models where the two-particle scattering matrix is a rational function depending on the difference of rapidities of particles involved in the scattering [16]. Yangian symmetry has been used to uniquely determine S -matrices describing the scattering of fundamental and bound-state magnons of closed spin chains [17, 18], and a remnant of it is expected to govern the scattering from the boundaries as well. Some charges conserved by this boundary symmetry have recently been constructed for open spin chains ending on giant graviton branes with broken symmetries [19]. It is an interesting challenge to understand the boundary symmetry in full: an important feature of quantum integrability is that the presence of suitable boundary conditions may break a bulk Yangian symmetry without spoiling integrability. Rather we expect to find a boundary symmetry which is a co-ideal subalgebra of the bulk symmetry [20, 21, 22], probably in the form of a generalized twisted Yangian [23].

Building on [19], our purpose here is to consider the general framework of boundary Yangian symmetry by considering the reflection of magnon bound states from the $D3$ brane known as the $Y = 0$ maximal giant graviton [24]. Depending on the choice of vacuum state and relative orientation of the graviton inside S^5 , the centrally extended $\mathfrak{psu}(2|2)$ symmetry algebra may be preserved by the boundary or broken down to $\mathfrak{su}(2|1)$ [24]. In this paper we construct the hidden boundary symmetry which extends $\mathfrak{su}(2|1)$, and find it to be of the form of the generalized twisted Yangians of [20, 21]. The two-magnon bound state reflection matrix which respects this symmetry proves to be in agreement with [19].

We also consider a toy-model boundary which breaks the symmetry down to $\mathfrak{su}(1|2)$. We show that it has a boundary Yangian symmetry of the same type as the $Y = 0$ giant graviton, but allows diagonal reflection only. Thus the reflection matrices for this case are fully determined by the boundary symmetry alone, and the boundary Yangian, although a nice mathematical example, is redundant, in contrast to earlier case.

This paper is organized as follows. In section 2 we briefly recall the superspace representation of the symmetry algebra and the Yangian symmetry of the bulk S -matrix. In section 3 we present the general framework for constructing a generalized twisted boundary Yangian, and construct the boundary remnant of the bulk Yangian symmetry for the $Y = 0$ giant graviton. Reflection from the boundary preserving the $\mathfrak{su}(1|2)$ subalgebra and its corresponding boundary Yangian and some complementary formulae are presented in the appendices.

2. Yangian symmetry of the S-matrix

We begin by briefly reviewing the centrally-extended $\mathfrak{psu}(2|2)$ algebra and its Yangian extension. This is the symmetry algebra of the excitations of the light-cone superstring theory on $AdS_5 \times S^5$ (and thereby of the S -matrix), and also of the single trace operators in the $\mathcal{N} = 4$ supersymmetric gauge theory that are analogous to, and known as, spin chains. We shall be using the superspace formalism introduced in [11], which simplifies greatly the calculations of the magnon bound state S - and K -matrices.

2.1 Superspace representation of the symmetry algebra

The centrally-extended $\mathfrak{psu}(2|2)$ has two sets of bosonic rotation generators $\mathbb{R}_a{}^b, \mathbb{L}_\alpha{}^\beta$, two sets of fermionic supersymmetry generators $\mathbb{Q}_\alpha{}^a, \mathbb{G}_a{}^\alpha$ and three central charges \mathbb{H}, \mathbb{C} and \mathbb{C}^\dagger . The non-trivial commutation relations are [5]

$$\begin{aligned} [\mathbb{L}_\alpha{}^\beta, \mathbb{J}_\gamma] &= \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2} \delta_\alpha^\beta \mathbb{J}_\gamma, & [\mathbb{L}_\alpha{}^\beta, \mathbb{J}^\gamma] &= -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2} \delta_\alpha^\beta \mathbb{J}^\gamma, \\ [\mathbb{R}_a{}^b, \mathbb{J}_c] &= \delta_c^b \mathbb{J}_a - \frac{1}{2} \delta_a^b \mathbb{J}_c, & [\mathbb{R}_a{}^b, \mathbb{J}^c] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2} \delta_a^b \mathbb{J}^c, \\ \{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} &= \epsilon^{ab} \epsilon_{\alpha\beta} \mathbb{C}, & \{\mathbb{G}_a{}^\alpha, \mathbb{G}_b{}^\beta\} &= \epsilon^{\alpha\beta} \epsilon_{ab} \mathbb{C}^\dagger, \\ \{\mathbb{Q}_\alpha{}^a, \mathbb{G}_b{}^\beta\} &= \delta_b^a \mathbb{L}_\beta{}^\alpha + \delta_\beta^\alpha \mathbb{R}_b{}^a + \delta_b^\alpha \delta_\beta^\alpha \mathbb{H}, \end{aligned} \quad (2.1)$$

where $a, b, \dots = 1, 2$ and $\alpha, \beta, \dots = 3, 4$.

A general l -magnon bound state is an atypical totally-symmetric representation of the centrally-extended $\mathfrak{psu}(2|2)$. The dimension of the representation is $2l|2l$ and may be neatly realized as degree l monomial on a graded vector space with the basis $\omega_1, \omega_2, \theta_3, \theta_4$, where ω_a and θ_α are bosonic and fermionic variables respectively [11].

In this representation the centrally-extended $\mathfrak{psu}(2|2)$ generators are realized as the differential operators

$$\begin{aligned} \mathbb{R}_a{}^b &= \omega_a \frac{\partial}{\partial \omega_b} - \frac{1}{2} \delta_a^b \omega_c \frac{\partial}{\partial \omega_c}, & \mathbb{L}_\alpha{}^\beta &= \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2} \delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial \theta_\gamma}, \\ \mathbb{Q}_\alpha{}^a &= a \theta_\alpha \frac{\partial}{\partial \omega_a} + b \epsilon^{ab} \epsilon_{\alpha\beta} \omega_b \frac{\partial}{\partial \theta_\beta}, & \mathbb{G}_a{}^\alpha &= c \epsilon_{ab} \epsilon^{\alpha\beta} \theta_\beta \frac{\partial}{\partial \omega_b} + d \omega_a \frac{\partial}{\partial \theta_\alpha}, \\ \mathbb{C} &= ab \left(\omega_a \frac{\partial}{\partial \omega_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), & \mathbb{C}^\dagger &= cd \left(\omega_a \frac{\partial}{\partial \omega_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), \\ \mathbb{H} &= (ad + bc) \left(\omega_a \frac{\partial}{\partial \omega_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), \end{aligned} \quad (2.2)$$

where the coefficients $a = a(p)$, $b = b(p)$, $c = c(p)$, $d = d(p)$ are the representation parameters and the corresponding vector space is denoted as $\mathcal{V}^l(p, \zeta)$, where p and ζ are complex parameters of the representation and correspond to the momentum and the phase of an individual magnon in the spin chain.

A convenient parametrization of the representation parameters is¹ [5]

$$a = \sqrt{\frac{g}{2l}}\eta, \quad b = \sqrt{\frac{g}{2l}}\frac{i\zeta}{\eta}\left(\frac{x^+}{x^-} - 1\right), \quad c = -\sqrt{\frac{g}{2l}}\frac{\eta}{\zeta x^+}, \quad d = -\sqrt{\frac{g}{2l}}\frac{x^+}{i\eta}\left(\frac{x^-}{x^+} - 1\right), \quad (2.3)$$

where g is a coupling constant, $\zeta = e^{2i\xi}$ is the magnon phase and x^\pm are the spectral parameters respecting the mass-shell (multiplet shortening) condition of the l -magnon bound state,

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = i\frac{2l}{g}. \quad (2.4)$$

Unitarity requires $\eta = e^{i\xi}e^{i\frac{\varphi}{2}}\sqrt{i(x^- - x^+)}$, where the arbitrary phase factor $e^{i\varphi}$ is parameterizing the freedom in choosing x^\pm .

The eigenvalues of the central charges of l -magnon bound state are expressed as

$$C_l = l ab = \frac{i}{2}g(e^{ip} - 1)e^{2i\xi}, \quad C_l^\dagger = l cd = -\frac{i}{2}g(e^{-ip} - 1)e^{-2i\xi}, \\ H_l = l(ad + bc) = \sqrt{l^2 + 4g^2 \sin^2 \frac{p}{2}}. \quad (2.5)$$

We write the S -matrix as a differential operator in superspace acting on the tensor product of two vector spaces

$$S(p_1, p_2) : \quad \mathcal{V}^M(p_1, \zeta) \otimes \mathcal{V}^N(p_2, \zeta e^{ip_1}) \rightarrow \mathcal{V}^M(p_1, \zeta e^{ip_2}) \otimes \mathcal{V}^N(p_2, \zeta), \quad (2.6)$$

where we have chosen phase ζ to increase from left to right. In the superspace formalism the S -matrix may be viewed as an element of

$$End(\mathcal{V}^M \otimes \mathcal{V}^N) \approx \mathcal{V}^M \otimes \mathcal{V}^N \otimes \mathcal{D}_M \otimes \mathcal{D}_N, \quad (2.7)$$

where \mathcal{D}_M is the vector space dual to \mathcal{V}^M . The dual vector space is realized as the space of polynomials of degree M of the differential operators $\frac{\partial}{\partial \omega_a}$ and $\frac{\partial}{\partial \theta_\alpha}$, with a natural pairing between \mathcal{D}_M and \mathcal{V}^M induced by the relations $\frac{\partial}{\partial \omega_a}\omega_b = \delta_b^a$ and $\frac{\partial}{\partial \theta_\alpha}\theta_\beta = \delta_\beta^\alpha$. Thus the S -matrix may be represented as

$$S(p_1, p_2) = \sum_i a_i(p_1, p_2) \Lambda_i, \quad (2.8)$$

where Λ_i span a complete basis of differential operators invariant under the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ algebra and $a_i(p_1, p_2)$ are S -matrix components. The exact expressions of Λ_i for various S -matrices and general formulas how to compute generic Λ_i are given in [11].

The invariance of the S -matrix under the co-products of the generators of the symmetry algebra reads as

$$S(p_1, p_2) \Delta(\mathbb{J}^A) = \Delta^{op}(\mathbb{J}^A) S(p_1, p_2), \quad (2.9)$$

¹There is a slight abuse of notation here, with a, b, c, d used both for the representation parameters and bosonic indices, but these are now the standard conventions. We shall avoid using these letters anywhere else!

where $\Delta^{op} = P\Delta P$ with P being a graded permutation. The invariance constrains all coefficients of the S -matrix of the fundamental states up to an overall phase. In the case of scattering of l - with m -bound states with $l, m \geq 2$, the symmetry algebra alone is not enough to fix all S -matrix coefficients, and additional constraints are required from the Yang-Baxter equation or, alternatively, Yangian symmetry [11, 18].

2.2 Yangian symmetry and co-products

The Yangian $Y(\mathfrak{g})$ of a Lie algebra \mathfrak{g} is a deformation of the universal enveloping algebra $U(\mathfrak{g}[u])$ of the polynomial algebra $\mathfrak{g}[u]$. It is generated by grade-0 \mathfrak{g} generators \mathbb{J}^A and grade-1 $Y(\mathfrak{g})$ generators $\hat{\mathbb{J}}^A$. Their commutators have the generic form

$$[\mathbb{J}^A, \mathbb{J}^B] = f^{AB}_C \mathbb{J}^C, \quad [\mathbb{J}^A, \hat{\mathbb{J}}^B] = f^{AB}_C \hat{\mathbb{J}}^C, \quad (2.10)$$

and must obey Jacobi and Serre relations

$$\begin{aligned} [\mathbb{J}^A, [\mathbb{J}^B, \mathbb{J}^C]] &= 0, & [\mathbb{J}^A, [\mathbb{J}^B, \hat{\mathbb{J}}^C]] &= 0, \\ [\hat{\mathbb{J}}^A, [\hat{\mathbb{J}}^B, \mathbb{J}^C]] &= \frac{1}{4} f^{AG}_D f^{BH}_E f^{CK}_F f_{GHK} \mathbb{J}^{\{D} \mathbb{J}^E \mathbb{J}^F\}. \end{aligned} \quad (2.11)$$

The indices of structure constants f^{AB}_D are lowered by the means of the inverse Killing-Cartan form g_{BD} . In the case of interest \mathfrak{g} is the centrally-extended $\mathfrak{psu}(2|2)$, and the relevant Killing form is degenerate. However, this degeneracy may be cured in several ways—for example, by considering the limit $\varepsilon \rightarrow 0$ of the exceptional superalgebra $\mathfrak{d}(2, 1; \varepsilon)$ [25], or by considering Drinfeld's second realization of the centrally-extended $\mathfrak{psu}(2|2)$, which was shown to be isomorphic to the first realization in [26].

The co-products of the grade-0 and grade-1 generators take form

$$\Delta \mathbb{J}^A = \mathbb{J}^A \otimes 1 + 1 \otimes \mathbb{J}^A, \quad \Delta \hat{\mathbb{J}}^A = \hat{\mathbb{J}}^A \otimes 1 + 1 \otimes \hat{\mathbb{J}}^A + \frac{1}{2} f^A_{BC} \mathbb{J}^B \otimes \mathbb{J}^C. \quad (2.12)$$

Crucial in constructing the finite-dimensional representations of $Y(\mathfrak{g})$ is the one-parameter family of the ‘evaluation automorphisms’

$$\tau_v : Y(\mathfrak{g}) \rightarrow Y(\mathfrak{g}) \quad \mathbb{J}^A \mapsto \mathbb{J}^A, \quad \hat{\mathbb{J}}^A \mapsto \hat{\mathbb{J}}^A + v \mathbb{J}^A, \quad (2.13)$$

corresponding to a shift in the polynomial variable, which implies that $Y(\mathfrak{g})$ representations appear in one-parameter families. On (the limited set of) finite-dimensional irreducible representations of \mathfrak{g} which may be extended to representations of $Y(\mathfrak{g})$, these are realized via the ‘evaluation map’

$$\text{ev}_v : Y(\mathfrak{g}) \mapsto U(\mathfrak{g}) \quad \mathbb{J}^A \mapsto \mathbb{J}^A, \quad \hat{\mathbb{J}}^A \mapsto v \mathbb{J}^A, \quad (2.14)$$

which yields ‘evaluation modules’, with states $|v\rangle$ carrying a spectral parameter v . As was shown in [12], the magnon states are of this form, and have

$$\hat{\mathbb{J}}^A |u\rangle = i \frac{g}{2} u \mathbb{J}^A |u\rangle, \quad (2.15)$$

where u is the rapidity of the corresponding magnon state. In the case of the bulk l -magnon bound states of the centrally-extended $\mathfrak{psu}(2|2)$, the rapidity is $u \equiv u(p) = x^+ + \frac{1}{x^+} - i\frac{1}{g}$.

The Yangian symmetry fixes the bound state S -matrices uniquely up to an overall phase by requiring their invariance under the co-products of the Yangian generators [12]

$$\begin{aligned}
\Delta \hat{\mathbb{R}}_a^b &= \hat{\mathbb{R}}_a^b \otimes 1 + 1 \otimes \hat{\mathbb{R}}_a^b + \frac{1}{2} \mathbb{R}_a^c \otimes \mathbb{R}_c^b - \frac{1}{2} \mathbb{R}_c^b \otimes \mathbb{R}_a^c - \frac{1}{2} \mathbb{G}_a^\gamma \otimes \mathbb{Q}_\gamma^b - \frac{1}{2} \mathbb{Q}_\gamma^b \otimes \mathbb{G}_a^\gamma \\
&\quad + \frac{1}{4} \delta_a^b \mathbb{G}_c^\gamma \otimes \mathbb{Q}_\gamma^c + \frac{1}{4} \delta_a^b \mathbb{Q}_\gamma^c \otimes \mathbb{G}_c^\gamma, \\
\Delta \hat{\mathbb{L}}_\alpha^\beta &= \hat{\mathbb{L}}_\alpha^\beta \otimes 1 + 1 \otimes \hat{\mathbb{L}}_\alpha^\beta - \frac{1}{2} \mathbb{L}_\alpha^\gamma \otimes \mathbb{L}_\gamma^\beta + \frac{1}{2} \mathbb{L}_\gamma^\beta \otimes \mathbb{L}_\alpha^\gamma + \frac{1}{2} \mathbb{G}_c^\beta \otimes \mathbb{Q}_\alpha^c + \frac{1}{2} \mathbb{Q}_\alpha^c \otimes \mathbb{G}_c^\beta \\
&\quad - \frac{1}{4} \delta_\alpha^\beta \mathbb{G}_c^\gamma \otimes \mathbb{Q}_\gamma^c - \frac{1}{4} \delta_\alpha^\beta \mathbb{Q}_\gamma^c \otimes \mathbb{G}_c^\gamma, \\
\Delta \hat{\mathbb{Q}}_\alpha^a &= \hat{\mathbb{Q}}_\alpha^a \otimes 1 + 1 \otimes \hat{\mathbb{Q}}_\alpha^a + \frac{1}{2} \mathbb{Q}_\alpha^c \otimes \mathbb{R}_c^a - \frac{1}{2} \mathbb{R}_c^a \otimes \mathbb{Q}_\alpha^c + \frac{1}{2} \mathbb{Q}_\gamma^a \otimes \mathbb{L}_\alpha^\gamma - \frac{1}{2} \mathbb{L}_\alpha^\gamma \otimes \mathbb{Q}_\gamma^a \\
&\quad + \frac{1}{4} \mathbb{Q}_\alpha^a \otimes \mathbb{H} - \frac{1}{4} \mathbb{H} \otimes \mathbb{Q}_\alpha^a + \frac{1}{2} \varepsilon_{\alpha\gamma} \varepsilon^{ac} \mathbb{C} \otimes \mathbb{G}_c^\gamma - \frac{1}{2} \varepsilon_{\alpha\gamma} \varepsilon^{ac} \mathbb{G}_c^\gamma \otimes \mathbb{C}, \\
\Delta \hat{\mathbb{G}}_a^\alpha &= \hat{\mathbb{G}}_a^\alpha \otimes 1 + 1 \otimes \hat{\mathbb{G}}_a^\alpha - \frac{1}{2} \mathbb{G}_c^\alpha \otimes \mathbb{R}_a^c + \frac{1}{2} \mathbb{R}_a^c \otimes \mathbb{G}_c^\alpha - \frac{1}{2} \mathbb{G}_a^\gamma \otimes \mathbb{L}_\gamma^\alpha + \frac{1}{2} \mathbb{L}_\gamma^\alpha \otimes \mathbb{G}_a^\gamma \\
&\quad - \frac{1}{4} \mathbb{G}_a^\alpha \otimes \mathbb{H} + \frac{1}{4} \mathbb{H} \otimes \mathbb{G}_a^\alpha - \frac{1}{2} \varepsilon_{ac} \varepsilon^{\alpha\gamma} \mathbb{C}^\dagger \otimes \mathbb{Q}_\gamma^c + \frac{1}{2} \varepsilon_{ac} \varepsilon^{\alpha\gamma} \mathbb{Q}_\gamma^c \otimes \mathbb{C}^\dagger, \\
\Delta \hat{\mathbb{C}} &= \hat{\mathbb{C}} \otimes 1 + 1 \otimes \hat{\mathbb{C}} - \frac{1}{2} \mathbb{H} \otimes \mathbb{C} + \frac{1}{2} \mathbb{C} \otimes \mathbb{H}, \\
\Delta \hat{\mathbb{C}}^\dagger &= \hat{\mathbb{C}}^\dagger \otimes 1 + 1 \otimes \hat{\mathbb{C}}^\dagger + \frac{1}{2} \mathbb{H} \otimes \mathbb{C}^\dagger - \frac{1}{2} \mathbb{C}^\dagger \otimes \mathbb{H}, \\
\Delta \hat{\mathbb{H}} &= \hat{\mathbb{H}} \otimes 1 + 1 \otimes \hat{\mathbb{H}} + \mathbb{C} \otimes \mathbb{C}^\dagger - \mathbb{C}^\dagger \otimes \mathbb{C}.
\end{aligned} \tag{2.16}$$

The non-trivial braiding factors are not explicitly shown in the co-products above—rather they are all hidden in the parameters of the representation (2.3) and the choice of phase in (2.6).

3. Boundary remnants of Yangian symmetry

In this section we shall present a method for constructing a boundary remnant of bulk Yangian symmetry, which takes the form of a generalized twisted Yangian $Y(\mathfrak{g}, \mathfrak{h})$, constructed as a subalgebra of $Y(\mathfrak{g})$. Here $\mathfrak{h} \subset \mathfrak{g}$ is the Lie subalgebra preserved by the boundary of the Lie algebra \mathfrak{g} respected by the bulk states [20, 21]. The construction requires that $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair (3.1) [27]. The centrally-extended algebra $\mathfrak{g} = \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ of the light-cone string may be split into a symmetric pair in two algebraically similar ways, with $\mathfrak{h} = \mathfrak{su}(2|1)$ and $\mathfrak{h} = \mathfrak{su}(1|2)$, but they are very different from the scattering theory point of view.

The first case, $\mathfrak{h} = \mathfrak{su}(2|1)$, corresponds to an open string ending on the $Y = 0$ giant graviton and leads to the reflection matrix governed by the boundary Yangian symmetry. We shall consider this case in the subsection 3.2.

The second case, $\mathfrak{h} = \mathfrak{su}(1|2)$, which we consider as a toy model, leads to a trivial (diagonal) reflection matrix fully determined by the symmetry algebra alone. Although it still possesses a boundary Yangian symmetry, this symmetry now appears redundant. We present it in the appendix A as a mathematical exercise; its role in the AdS/CFT correspondence is not clear.

3.1 Boundary Yangian symmetry

Consider an integrable boundary field theory which preserves only a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the symmetry respected by the bulk fields. We want to find the corresponding Yangian charges conserved under the reflection. For this purpose, we shall be considering the boundary Yangian algebra acting on the evaluation module (2.15) carrying the spectral parameter u which is mapped to the rapidity of the state in field theory. The reflection in the integrable field theory results in a change of sign of the rapidity $u \mapsto -u$.

Integrability requires that the boundary Lie symmetry $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra invariant under an involution σ [27]. Thus we proceed by splitting the bulk algebra into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ under a graded involution σ of \mathfrak{g} with the eigenspaces $\sigma(\mathfrak{h}) = +1$ and $\sigma(\mathfrak{m}) = -1$. One then has

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}, \quad (3.1)$$

together with orthogonality with respect to the Killing form, $\kappa(\mathfrak{h}, \mathfrak{m}) = 0$. This is crucial in guaranteeing the co-ideal property – that the co-product of any Yangian charge $\hat{\mathbb{J}}$ preserved by the boundary must be in the tensor product of bulk and boundary Yangian

$$\Delta \hat{\mathbb{J}} \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h}). \quad (3.2)$$

This ensures that multiparticle products of bulk and boundary states still represent $Y(\mathfrak{g}, \mathfrak{h})$, and is analogous to the requirement that Δ be a homomorphism $Y(\mathfrak{g}) \rightarrow Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$ imposed by multiparticle bulk states.

Just as $Y(\mathfrak{g})$ was a deformation of $U(\mathfrak{g}[u])$, so $Y(\mathfrak{g}, \mathfrak{h})$ may be thought of as a deformation of the subalgebra of $U(\mathfrak{g}[u])$ which is invariant under the extension $\bar{\sigma}$ of σ which sends $\bar{\sigma} : u \mapsto -u$ (representing the change of sign of the magnon rapidity after the reflection),

$$\mathfrak{h} \oplus u\mathfrak{m} \oplus \dots \subset \mathfrak{g}[u] = (\mathfrak{h} \oplus \mathfrak{m}) \oplus u(\mathfrak{h} \oplus \mathfrak{m}) \oplus \dots \quad (3.3)$$

Hence the boundary Yangian charges must live in the subspace $u\mathfrak{m}$. However, while the grade-0 generators of \mathfrak{h} clearly respect the co-ideal property, the grade-1 generators of $u\mathfrak{m}$ do not do so,

$$\Delta \hat{\mathbb{J}}^p = \hat{\mathbb{J}}^p \otimes 1 + 1 \otimes \hat{\mathbb{J}}^p + \frac{1}{2} f_{qi}^p (\mathbb{J}^q \otimes \mathbb{J}^i + \mathbb{J}^i \otimes \mathbb{J}^q) \notin Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h}), \quad (3.4)$$

where i, j, k, \dots run over the \mathfrak{h} -indices and p, q, r, \dots over the \mathfrak{m} -indices. Rather we need a deformation of the grade-1 $u\mathfrak{m}$ generators, and therefore we find $Y(\mathfrak{g}, \mathfrak{h})$ to be the algebra

generated by $\{\mathbb{J}^i, \tilde{\mathbb{J}}^p\}$, where

$$\tilde{\mathbb{J}}^p := \hat{\mathbb{J}}^p + \frac{1}{2} f_{qi}^p \mathbb{J}^q \mathbb{J}^i, \quad (3.5)$$

are the twisted boundary Yangian generators. (One can arrive at the same deformation in different ways — for example, via the boundary transfer matrix, or by considering the charges' classical conservation [21].)

Now we can show that $Y(\mathfrak{g}, \mathfrak{h})$ is a left co-ideal subalgebra, $\Delta Y(\mathfrak{g}, \mathfrak{h}) \subset Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h})$. To do this one calculates explicitly the co-product of the twisted Yangian generator,

$$\begin{aligned} \Delta \tilde{\mathbb{J}}^p &= \Delta \hat{\mathbb{J}}^p + \frac{1}{2} f_{qi}^p \Delta \mathbb{J}^q \Delta \mathbb{J}^i \\ &= \hat{\mathbb{J}}^p \otimes 1 + 1 \otimes \hat{\mathbb{J}}^p + \frac{1}{2} f_{qi}^p (\mathbb{J}^q \mathbb{J}^i \otimes 1 + 1 \otimes \mathbb{J}^q \mathbb{J}^i) \\ &\quad + \frac{1}{2} f_{iq}^p \mathbb{J}^i \otimes \mathbb{J}^q + \frac{1}{2} f_{qi}^p \mathbb{J}^q \otimes \mathbb{J}^i + \frac{1}{2} f_{qi}^p (\mathbb{J}^q \otimes \mathbb{J}^i + \mathbb{J}^i \otimes \mathbb{J}^q) \\ &= \tilde{\mathbb{J}}^p \otimes 1 + 1 \otimes \tilde{\mathbb{J}}^p + f_{qi}^p \mathbb{J}^q \otimes \mathbb{J}^i \\ &\in Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h}), \end{aligned} \quad (3.6)$$

where we have used the implication of the symmetric pair decomposition (3.1) that the only non-zero structure constants are f^{pqi} and f^{pij} .

We should emphasize that the map $\bar{\sigma}$ together with the twist (3.5) is an automorphism of $\mathfrak{g}[u]$ realized on the ‘evaluation module’ (2.15) and is independent of its explicit realization in field theory, *i.e.* it is *not* a map on the fields.

3.2 Yangian symmetry of the $Y=0$ maximal giant graviton

Maximal giant gravitons are $D3$ branes in $AdS_5 \times S^5$ which wraps a topologically-trivial cycles comprising a maximal S^3 within the S^5 . Giant gravitons are prevented from collapsing by their coupling to the background supergravity fields. The usual parametrization of the S^5 is expressed in terms of $X = \Phi_1 + i\Phi_2$, $Y = \Phi_3 + i\Phi_4$, $Z = \Phi_5 + i\Phi_6$ respecting $|X|^2 + |Y|^2 + |Z|^2 = 1$, where the radius of the S^5 has been normalized to $R = 1$. In this parametrization the maximal giant graviton may be obtained by setting any two Φ_i to zero.

Any two such configurations are of course related by an $SO(6)$ rotation. However, one can break this equivalence by attaching an open string to the brane and giving the string a charge J corresponding to a preferred $SO(2) \subset SO(6)$ rotation. In the limit when J is large the field theory description of the string carries a large number of insertions of the field corresponding to the preferred rotation. It was shown in [24] that the explicit description of the open string depends on the selection of a particular generator J and the relevant orientation of the giant graviton inside S^5 . The two interesting cases are given by choosing the charge to be $J = J_{56}$ and the giant graviton to be a three sphere given by $Y = 0$ or $Z = 0$.

The $Y = 0$ giant graviton preserves the subgroup which is also preserved by the field Y . This restricts the symmetry algebra on the boundary to be $\mathfrak{h} = \mathfrak{su}(2|1)$ and has no degrees of freedom attached to the end of the spin chain [24]. The commutation relations of $\mathfrak{su}(2|1)$ are acquired from (2.1) by dropping the generators with bosonic indices $a, b, c, \dots = 2$; thus the surviving generators are $\mathbb{L}_\alpha^\beta, \mathbb{R}_1^1, \mathbb{R}_2^2 \equiv -\mathbb{R}_1^1, \mathbb{Q}_\alpha^1, \mathbb{G}_1^\alpha$ and \mathbb{H} . It is straightforward to check that the subalgebra $\mathfrak{h} = \mathfrak{su}(2|1)$ and subset $\mathfrak{m} = \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3 / \mathfrak{su}(2|1)$ consisting of generators $\mathbb{R}_1^2, \mathbb{R}_2^1, \mathbb{Q}_\gamma^2, \mathbb{G}_2^\gamma, \mathbb{C}, \mathbb{C}^\dagger$ form a symmetric pair (3.1) by considering the commutation relations (2.1). Hence the theory should possess a boundary Yangian.

We shall construct the scattering theory first. Following [19] we define a boundary vacuum state $|0_B\rangle$ and a corresponding trivial vector space $\mathcal{V}_B(0)$ which is annihilated by all $\mathfrak{su}(2|1)$ generators. This allows us to define the superspace K -matrix for the reflection of bulk magnons from the boundary vacuum state as an operator acting on the tensor product of spaces,

$$K(p) : \mathcal{V}^M(p, \zeta) \otimes \mathcal{V}_B(0) \rightarrow \mathcal{V}^M(-p, \zeta) \otimes \mathcal{V}_B(0), \quad (3.7)$$

where the reflection matrix is defined as a differential operator

$$K(p) = \sum_i k_i(p) \Lambda_i \quad (3.8)$$

acting on the superspace. In the case of the reflection of fundamental states, the symmetry algebra implies that the only dynamics allowed is $\omega_i \rightarrow \omega_i$ and $\theta_\alpha \rightarrow \theta_\alpha$ with different reflection coefficients for ω_1 and ω_2 , but the same for θ_3 and θ_4 . Hence the reflection matrix K^A may be represented on a superspace as

$$K^A(p) = k_1(p) \omega_1 \frac{\partial}{\partial \omega_1} + k_2(p) \omega_2 \frac{\partial}{\partial \omega_2} + k_3(p) \theta_\alpha \frac{\partial}{\partial \theta_\alpha}. \quad (3.9)$$

The boundary symmetry algebra fixes the reflection coefficients (up to an overall factor) to be

$$k_1(p) = 1, \quad k_2(p) = -\frac{x^-}{x^+}, \quad k_3(p) = \frac{\tilde{\eta}}{\eta}. \quad (3.10)$$

In the case of the reflection of two-magnon bound states, reflection matrix K^B is no longer diagonal. The diagonal reflection channels are $\omega_i \omega_j \rightarrow \omega_i \omega_j$, $\omega_i \theta_\alpha \rightarrow \omega_i \theta_\alpha$, $\theta_3 \theta_4 \rightarrow \theta_3 \theta_4$, and the off-diagonal are $\omega_1 \omega_2 \rightarrow \theta_3 \theta_4$ and $\theta_3 \theta_4 \rightarrow \omega_1 \omega_2$. This leads to the following representation of the reflection matrix on the superspace

$$K^B(p) = \sum_{i=1}^8 k_i(p) \Lambda_i, \quad (3.11)$$

where Λ_i with $i = 1, \dots, 6$ are diagonal and Λ_7, Λ_8 are off-diagonal differential operators

$$\begin{aligned} \Lambda_1 &= \frac{1}{2} \omega_1 \omega_1 \frac{\partial^2}{\partial \omega_1 \partial \omega_1}, & \Lambda_2 &= \omega_1 \omega_2 \frac{\partial^2}{\partial \omega_2 \partial \omega_1}, & \Lambda_3 &= \frac{1}{2} \omega_2 \omega_2 \frac{\partial^2}{\partial \omega_2 \partial \omega_2}, \\ \Lambda_4 &= \theta_3 \theta_4 \frac{\partial^2}{\partial \theta_4 \partial \theta_3}, & \Lambda_5 &= \omega_1 \theta_\alpha \frac{\partial^2}{\partial \omega_1 \partial \theta_\alpha}, & \Lambda_6 &= \omega_2 \theta_\alpha \frac{\partial^2}{\partial \omega_2 \partial \theta_\alpha}, \\ \Lambda_7 &= \theta_3 \theta_4 \frac{\partial^2}{\partial \omega_2 \partial \omega_1}, & \Lambda_8 &= \omega_1 \omega_2 \frac{\partial^2}{\partial \theta_4 \partial \theta_3}. \end{aligned} \quad (3.12)$$

In this case, following the general pattern [11, 18], the symmetry algebra alone is not enough to fix all reflection coefficients uniquely. This is the consequence of the relation between representations of $\mathfrak{su}(2|2)$ and $\mathfrak{su}(2|1)$. Fundamental magnons transform irreducibly in the fundamental representation \square of $\mathfrak{su}(2|2)$ and in the supersymmetric representation $\square\square$ of $\mathfrak{su}(2|1)$. This is no longer the case for the two-magnon bound states, which transform irreducibly in a supersymmetric representation $\square\square$ of $\mathfrak{su}(2|2)$, but in a *reducible* representation $\square\square\square\square \oplus \square\square$ of $\mathfrak{su}(2|1)$: and one further needs either the boundary Yang-Baxter equation or boundary Yangian symmetry to fix the ratio between the representations of $\mathfrak{su}(2|1)$. The number of reducible components grows with bound state number l . In the case of reflection of two-magnon bound states the symmetry algebra fixes 7 out of 8 reflection coefficients up to an overall dressing phase. Hence one needs to impose only one additional constraint to fix the last coefficient. A conserved Yangian charge $\tilde{\mathbb{Q}}$ giving the required constraint was constructed and the reflection matrix K^B was calculated in [19] — but one charge alone would not typically be enough to constrain uniquely the higher-order bound state S -matrices.

We shall construct the boundary Yangian $Y(\mathfrak{g}, \mathfrak{h})$ using (3.5). The co-product of twisted Yangian generators $\Delta\tilde{\mathbb{J}}^p$ acts on the tensor product of bulk and boundary vector spaces of bulk and boundary algebra and, by the construction above, any generator of the boundary symmetry annihilates the boundary vacuum state $|0_B\rangle$. Hence the non-trivial parts of the co-products of twisted Yangian generators are

$$\begin{aligned}
\Delta\tilde{\mathbb{R}}_1^2 &= \left(\hat{\mathbb{R}}_1^2 + \frac{1}{2}\mathbb{R}_1^2\mathbb{R}_1^1 - \frac{1}{2}\mathbb{R}_1^2\mathbb{R}_2^2 - \frac{1}{2}\mathbb{Q}_\gamma^2\mathbb{G}_1^\gamma \right) \otimes 1, \\
\Delta\tilde{\mathbb{R}}_2^1 &= \left(\hat{\mathbb{R}}_2^1 + \frac{1}{2}\mathbb{R}_2^1\mathbb{R}_1^1 - \frac{1}{2}\mathbb{R}_2^1\mathbb{R}_2^2 - \frac{1}{2}\mathbb{G}_2^\gamma\mathbb{Q}_\gamma^1 \right) \otimes 1, \\
\Delta\tilde{\mathbb{Q}}_\alpha^2 &= \left(\hat{\mathbb{Q}}_\alpha^2 + \frac{1}{2}\mathbb{Q}_\alpha^2\mathbb{R}_2^2 - \frac{1}{2}\mathbb{R}_1^2\mathbb{Q}_\alpha^1 + \frac{1}{2}\mathbb{Q}_\gamma^2\mathbb{L}_\alpha^\gamma + \frac{1}{4}\mathbb{Q}_\alpha^2\mathbb{H} - \frac{1}{2}\varepsilon_{\alpha\gamma}\mathbb{C}\mathbb{G}_1^\gamma \right) \otimes 1, \\
\Delta\tilde{\mathbb{G}}_2^\alpha &= \left(\hat{\mathbb{G}}_2^\alpha - \frac{1}{2}\mathbb{G}_2^\alpha\mathbb{R}_2^2 + \frac{1}{2}\mathbb{R}_2^1\mathbb{G}_1^\alpha - \frac{1}{2}\mathbb{G}_2^\gamma\mathbb{L}_\gamma^\alpha - \frac{1}{4}\mathbb{G}_2^\alpha\mathbb{H} + \frac{1}{2}\varepsilon^{\alpha\gamma}\mathbb{C}^\dagger\mathbb{Q}_\gamma^1 \right) \otimes 1, \\
\Delta\tilde{\mathbb{C}} &= \left(\hat{\mathbb{C}} + \frac{1}{2}\mathbb{C}\mathbb{H} \right) \otimes 1, \\
\Delta\tilde{\mathbb{C}}^\dagger &= \left(\hat{\mathbb{C}}^\dagger - \frac{1}{2}\mathbb{C}^\dagger\mathbb{H} \right) \otimes 1.
\end{aligned} \tag{3.13}$$

The braiding factors are defined by the reflection equation (3.7)² and are hidden in the representation parameters (2.3). The full expressions of the co-products is given in the appendix B. As expected, the co-product of the twisted Yangian generator $\tilde{\mathbb{R}}_2^1$ coincides with the conserved charge $\tilde{\mathbb{Q}}$ of [19].

²The braiding for the left factor of the co-products is always trivial because the reflection results only in a change of sign for momentum $p \mapsto -p$ of the incoming magnon, *i.e.* maps spectral parameters $x^\pm \mapsto -x^\mp$. The braiding would be non-trivial in the right factor of co-products corresponding to boundaries with degrees of freedom.

Requiring that the reflection matrix respect the co-products (3.13),

$$K(p) \Delta(\tilde{\mathbb{J}}^p) - \Delta(\tilde{\mathbb{J}}^p) K(p) = 0, \quad (3.14)$$

one finds the reflection matrix K^B coefficients to be

$$\begin{aligned} k_1(p) &= 1, & k_2(p) &= -\frac{y^- + y^- (y^+)^2}{y^+ + y^- (y^+)^2}, & k_3(p) &= \frac{y^-}{y^+}, \\ k_4(p) &= \frac{(1 + (y^-)^2) y^+ \tilde{\eta}^2}{y^- (1 + y^- y^+) \eta^2}, & k_5(p) &= \frac{\tilde{\eta}}{\eta}, & k_6(p) &= -\frac{\tilde{\eta}}{\eta}, \\ k_7(p) &= \frac{i \tilde{\eta}^2}{\zeta (1 + y^- y^+)}, & k_8(p) &= \frac{i \zeta}{(1 + y^- y^+) \eta^2}. \end{aligned} \quad (3.15)$$

It is worthwhile to note that, from the scattering-theory point of view, scattering from the $Y = 0$ giant graviton is identical to that from the $Y = 0$ $D7$ brane [28]. Consequently the reflection matrices and boundary Yangian are the same.

4. Discussion

In this paper we have constructed a boundary remnant of Yangian symmetry for the $Y = 0$ giant graviton, as a special case of the generalized twisted Yangian boundary symmetries $Y(\mathfrak{g}, \mathfrak{h})$ of [21].

We have shown that the reflection matrices of the $Y = 0$ giant graviton, which preserves only an $\mathfrak{su}(2|1)$ subalgebra of the centrally-extended $\mathfrak{psu}(2|2)$ algebra and has no boundary degrees of freedom, respect a boundary remnant $Y(\mathfrak{g}, \mathfrak{h})$ of the bulk Yangian symmetry $Y(\mathfrak{g})$ [12] which extends the results of [19]: the conserved charge constructed in [19] is a generator of the twisted boundary Yangian $Y(\mathfrak{g}, \mathfrak{h})$ we have constructed. Furthermore, from the scattering theory point of view, the reflection from the $Y = 0$ giant graviton is equivalent to reflection from the $Y = 0$ $D7$ brane [28] and the corresponding reflection matrices and Yangian symmetry are the same.

We have also considered the Yangian symmetry of a boundary which preserves an $\mathfrak{su}(1|2)$ subalgebra of the centrally-extended $\mathfrak{su}(2|2)$ algebra, with no boundary degrees of freedom. We showed that this leads to a diagonal reflection matrix for all l -magnon bound states and possesses a boundary Yangian. However, this hidden symmetry is redundant, because the reflection matrices are fully determined by the Lie symmetry algebra alone. The meaning of this in the AdS/CFT correspondence is not clear; we present it (in an appendix) merely as a nice example in a contrast to the $Y = 0$ giant graviton. One could try to consider a boundary identical to the $Z = 0$ giant graviton but with no boundary degrees of freedom, but it is easy to check that this kind of configuration is ruled out by the $\mathfrak{su}(2|2)$ algebra.

We have thus made some progress towards a better general understanding of boundary symmetry and boundary states in AdS/CFT. There is one further case of boundary symmetry which is not present in the case of the giant graviton, but is present in the left factor of the reflection from the $Z = 0$ $D7$ brane [28]. This preserves neither supersymmetries nor boundary degrees of freedom, and the preserved algebra is not part of a symmetric pair. A very similar reflection structure was addressed in [29], and we hope to understand this reflection problem in near future.

It is worth noting that the fundamental S -matrix possesses a secret symmetry [30] which emerges from the exceptional superalgebra $\mathfrak{d}(2, 1; \varepsilon)$ in the limit $\varepsilon \rightarrow 0$ [25]. We expect the boundary Yangian to be rich in such secrets too.

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A. Yangian symmetry of the $\mathfrak{su}(1|2)$ boundary

We consider a spin chain ending on a boundary which preserves only a $\mathfrak{h} = \mathfrak{su}(1|2)$ subalgebra of the bulk symmetry, and which has no degrees of freedom attached to the end of the spin chain. The commutation relations of $\mathfrak{su}(1|2)$ are inherited from the (2.1) by dropping the generators with fermionic indices $\alpha, \beta, \gamma, \dots = 4$; thus the surviving generators are $\mathbb{R}_a^b, \mathbb{L}_3^3 \equiv -\mathbb{L}_4^4, \mathbb{Q}_3^a, \mathbb{G}_a^3$ and \mathbb{H} . It is straightforward to check that the subalgebra $\mathfrak{h} = \mathfrak{su}(1|2)$ and subset $\mathfrak{m} = \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3/\mathfrak{su}(1|2)$ consisting of generators $\mathbb{L}_3^4, \mathbb{L}_4^3, \mathbb{Q}_4^a, \mathbb{G}_a^4$ and central charges $\mathbb{C}, \mathbb{C}^\dagger$ form a symmetric pair (3.1) by considering the commutation relations in the (2.1). Thus the theory should possess a boundary Yangian with the same structure as (3.13). Using the general prescription (3.5) one finds the non-trivial part of the boundary Yangian co-products to be

$$\begin{aligned}
\Delta \tilde{\mathbb{L}}_3^4 &= \left(\hat{\mathbb{L}}_3^4 - \frac{1}{2} \mathbb{L}_3^4 \mathbb{L}_4^4 + \frac{1}{2} \mathbb{L}_3^4 \mathbb{L}_3^3 + \frac{1}{2} \mathbb{G}_c^4 \mathbb{Q}_3^c \right) \otimes 1, \\
\Delta \tilde{\mathbb{L}}_4^3 &= \left(\hat{\mathbb{L}}_4^3 - \frac{1}{2} \mathbb{L}_4^3 \mathbb{L}_3^3 + \frac{1}{2} \mathbb{L}_4^3 \mathbb{L}_4^4 + \frac{1}{2} \mathbb{Q}_4^c \mathbb{G}_c^3 \right) \otimes 1, \\
\Delta \tilde{\mathbb{Q}}_4^a &= \left(\hat{\mathbb{Q}}_4^a + \frac{1}{2} \mathbb{Q}_4^c \mathbb{R}_c^a + \frac{1}{2} \mathbb{Q}_4^a \mathbb{L}_4^4 - \frac{1}{2} \mathbb{L}_4^3 \mathbb{Q}_3^a + \frac{1}{4} \mathbb{Q}_4^a \mathbb{H} - \frac{1}{2} \varepsilon^{ad} \mathbb{C} \mathbb{G}_d^3 \right) \otimes 1, \\
\Delta \tilde{\mathbb{G}}_a^4 &= \left(\hat{\mathbb{G}}_a^4 - \frac{1}{2} \mathbb{G}_c^4 \mathbb{R}_a^c - \frac{1}{2} \mathbb{G}_a^4 \mathbb{L}_4^4 + \frac{1}{2} \mathbb{L}_3^4 \mathbb{G}_a^3 - \frac{1}{4} \mathbb{G}_a^\alpha \mathbb{H} + \frac{1}{2} \varepsilon_{ac} \mathbb{C}^\dagger \mathbb{Q}_3^c \right) \otimes 1, \\
\Delta \tilde{\mathbb{C}} &= \left(\hat{\mathbb{C}} + \frac{1}{2} \mathbb{C} \mathbb{H} \right) \otimes 1, \\
\Delta \tilde{\mathbb{C}}^\dagger &= \left(\hat{\mathbb{C}}^\dagger - \frac{1}{2} \mathbb{C}^\dagger \mathbb{H} \right) \otimes 1.
\end{aligned} \tag{A.1}$$

Once again, the full expressions of the co-products are presented in the appendix B.

We shall construct a scattering theory in the same way as was done in subsection 2.2 for the $Y = 0$ giant graviton. First, we introduce a boundary vacuum state $|0_B\rangle$ and a corresponding trivial vector space $\mathcal{V}(0)$ which is annihilated by all $\mathfrak{su}(1|2)$ generators. This construction leads to the superspace K -matrix for the reflection of bulk magnons from the boundary vacuum state as an operator acting on the tensor product

$$K(p) : \mathcal{V}^M(p, \zeta) \otimes \mathcal{V}_B(0) \rightarrow \mathcal{V}^M(-p, \zeta) \otimes \mathcal{V}_B(0), \quad (\text{A.2})$$

where the reflection matrix is defined as a differential operator

$$K(p) = \sum_i k_i(p) \Lambda_i \quad (\text{A.3})$$

acting on the superspace. In the case of the reflection of fundamental states, following the similar considerations as for the $\mathfrak{su}(2|1)$ case, the symmetry algebra implies that the reflection matrix K^A is a diagonal matrix

$$K^A(p) = k_1(p) \omega_a \frac{\partial}{\partial \omega_a} + k_2(p) \theta_3 \frac{\partial}{\partial \theta_3} + k_3(p) \theta_4 \frac{\partial}{\partial \theta_4}. \quad (\text{A.4})$$

Then the boundary symmetry algebra fixes the reflection coefficients (up to an overall factor) to be

$$k_1(p) = 1, \quad k_2(p) = \frac{\tilde{\eta}}{\eta}, \quad k_3(p) = -\frac{x^+ \tilde{\eta}}{x^- \eta}. \quad (\text{A.5})$$

In the case of the reflection of two-magnon bound states, the most general structure of the reflection matrix K^B one may write is

$$K^B(p) = \sum_{i=1}^6 k_i(p) \Lambda_i, \quad (\text{A.6})$$

where Λ_i with $i = 1, \dots, 4$ are diagonal and Λ_5, Λ_6 are off-diagonal differential operators

$$\begin{aligned} \Lambda_1 &= \frac{1}{2} \omega_b \omega_a \frac{\partial^2}{\partial \omega_b \partial \omega_a}, & \Lambda_2 &= \omega_a \theta_3 \frac{\partial^2}{\partial \omega_a \partial \theta_3}, & \Lambda_3 &= \omega_a \theta_4 \frac{\partial^2}{\partial \omega_a \partial \theta_4}, \\ \Lambda_4 &= \theta_3 \theta_4 \frac{\partial^2}{\partial \theta_4 \partial \theta_3}, & \Lambda_5 &= \theta_3 \theta_4 \frac{\partial^2}{\partial \omega_2 \partial \omega_1}, & \Lambda_6 &= \omega_1 \omega_2 \frac{\partial^2}{\partial \theta_4 \partial \theta_3}. \end{aligned} \quad (\text{A.7})$$

However, the off-diagonal reflection channels are forbidden by the boundary symmetry. It is easy to see this by considering the invariance of the K -matrix under the R symmetry generator \mathbb{R}_2^1

$$\mathbb{R}_2^1 K^B \omega_1 \omega_1 = 2 k_1 \omega_1 \omega_2, \quad K^B \mathbb{R}_2^1 \omega_1 \omega_1 = 2 k_1 \omega_1 \omega_2 + 2 k_5 \theta_3 \theta_4, \quad (\text{A.8})$$

leading to $k_5 = 0$, and

$$\mathbb{R}_2^1 K^B \theta_3 \theta_4 = k_6 \omega_2 \omega_2, \quad K^B \mathbb{R}_2^1 \theta_3 \theta_4 = 0, \quad (\text{A.9})$$

leading to $k_6 = 0$. This is a general feature for the reflection of any l -magnon bound states. Hence the general l -magnon reflection matrix

$$K^l(p) = \sum_{i=1}^4 k_i(p) \Lambda_i \quad (\text{A.10})$$

is a diagonal matrix with

$$\begin{aligned}\Lambda_1 &= \frac{1}{l!} \omega_a^l \frac{\partial^l}{\partial \omega_a^l}, & \Lambda_2 &= \frac{1}{(l-1)!} \omega_a^{l-1} \theta_3 \frac{\partial^l}{\partial \omega_a^{l-1} \partial \theta_3}, \\ \Lambda_3 &= \frac{1}{(l-1)!} \omega_a^{l-1} \theta_4 \frac{\partial^l}{\partial \omega_a^{l-1} \partial \theta_4}, & \Lambda_4 &= \frac{1}{(l-2)!} \omega_a^{l-2} \theta_3 \theta_4 \frac{\partial^l}{\partial \omega_a^{l-2} \partial \theta_3 \partial \theta_4}.\end{aligned}\quad (\text{A.11})$$

The boundary symmetry algebra fixes the reflection coefficients uniquely (up to an overall factor) without need of the boundary Yangian symmetry. They are

$$k_1(p) = 1, \quad k_2(p) = \frac{\tilde{\eta}}{\eta}, \quad k_3(p) = -\frac{y^+ \tilde{\eta}}{y^- \eta}, \quad k_4(p) = -\frac{y^+ \tilde{\eta}^2}{y^- \eta^2}, \quad (\text{A.12})$$

here y^\pm are the spectral parameters of the l -magnon bound state. On the other hand one could have used the boundary Yangian (A.1) instead, leading to the same results.

This case is somewhat similar to the left factor of the reflection from the $Z = 0$ $D7$ brane [28, 31], which preserves neither supersymmetries nor boundary degrees of freedom and allows only diagonal reflection matrices K^l identical for any $l \geq 2$. There, the absence of supersymmetries preserved by the boundary required that the boundary Yang-Baxter equation be used to determine the reflection matrices.

B. The co-products of the boundary Yangian $Y(\mathfrak{g}, \mathfrak{h})$

The complete expressions of the co-products of the boundary twisted Yangian $Y(\mathfrak{g}, \mathfrak{h})$ defined by the algebra $\mathfrak{g} = \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ and subalgebra $\mathfrak{h} = \mathfrak{su}(2|1)$ preserved on the boundary are

$$\begin{aligned}\Delta \tilde{\mathbb{R}}_1^2 &= \left(\hat{\mathbb{R}}_1^2 + \frac{1}{2} \mathbb{R}_1^2 \mathbb{R}_1^1 - \frac{1}{2} \mathbb{R}_1^2 \mathbb{R}_2^2 - \frac{1}{2} \mathbb{Q}_\gamma^2 \mathbb{G}_1^\gamma \right) \otimes 1 \\ &\quad + 1 \otimes \left(\hat{\mathbb{R}}_1^2 + \frac{1}{2} \mathbb{R}_1^2 \mathbb{R}_1^1 - \frac{1}{2} \mathbb{R}_1^2 \mathbb{R}_2^2 - \frac{1}{2} \mathbb{Q}_\gamma^2 \mathbb{G}_1^\gamma \right) \\ &\quad + \mathbb{R}_1^2 \otimes \mathbb{R}_1^1 - \mathbb{R}_1^2 \otimes \mathbb{R}_2^2 - \mathbb{Q}_\gamma^2 \otimes \mathbb{G}_1^\gamma, \\ \Delta \tilde{\mathbb{R}}_2^1 &= \left(\hat{\mathbb{R}}_2^1 + \frac{1}{2} \mathbb{R}_2^1 \mathbb{R}_1^1 - \frac{1}{2} \mathbb{R}_2^1 \mathbb{R}_2^2 - \frac{1}{2} \mathbb{G}_2^\gamma \mathbb{Q}_\gamma^1 \right) \otimes 1 \\ &\quad + 1 \otimes \left(\hat{\mathbb{R}}_2^1 + \frac{1}{2} \mathbb{R}_2^1 \mathbb{R}_1^1 - \frac{1}{2} \mathbb{R}_2^1 \mathbb{R}_2^2 - \frac{1}{2} \mathbb{G}_2^\gamma \mathbb{Q}_\gamma^1 \right) \\ &\quad + \mathbb{R}_2^1 \otimes \mathbb{R}_1^1 - \mathbb{R}_2^1 \otimes \mathbb{R}_2^2 - \mathbb{G}_2^\gamma \otimes \mathbb{Q}_\gamma^1, \\ \Delta \tilde{\mathbb{Q}}_\alpha^2 &= \left(\hat{\mathbb{Q}}_\alpha^2 + \frac{1}{2} \mathbb{Q}_\alpha^2 \mathbb{R}_2^2 - \frac{1}{2} \mathbb{R}_1^2 \mathbb{Q}_\alpha^1 + \frac{1}{2} \mathbb{Q}_\gamma^2 \mathbb{L}_\alpha^\gamma + \frac{1}{4} \mathbb{Q}_\alpha^2 \mathbb{H} - \frac{1}{2} \varepsilon_{\alpha\gamma} \mathbb{C} \mathbb{G}_1^\gamma \right) \otimes 1 \\ &\quad + 1 \otimes \left(\hat{\mathbb{Q}}_\alpha^2 + \frac{1}{2} \mathbb{Q}_\alpha^2 \mathbb{R}_2^2 - \frac{1}{2} \mathbb{R}_1^2 \mathbb{Q}_\alpha^1 + \frac{1}{2} \mathbb{Q}_\gamma^2 \mathbb{L}_\alpha^\gamma + \frac{1}{4} \mathbb{Q}_\alpha^2 \mathbb{H} - \frac{1}{2} \varepsilon_{\alpha\gamma} \mathbb{C} \mathbb{G}_1^\gamma \right) \\ &\quad + \mathbb{Q}_\alpha^2 \otimes \mathbb{R}_2^2 - \mathbb{R}_1^2 \otimes \mathbb{Q}_\alpha^1 + \mathbb{Q}_\gamma^2 \otimes \mathbb{L}_\alpha^\gamma + \frac{1}{2} \mathbb{Q}_\alpha^2 \otimes \mathbb{H} - \varepsilon_{\alpha\gamma} \mathbb{C} \otimes \mathbb{G}_1^\gamma,\end{aligned}$$

$$\begin{aligned}
\Delta \tilde{G}_2^\alpha &= \left(\hat{G}_2^\alpha - \frac{1}{2} G_2^\alpha \mathbb{R}_2^2 + \frac{1}{2} \mathbb{R}_2^1 G_1^\alpha - \frac{1}{2} G_2^\gamma \mathbb{L}_\gamma^\alpha - \frac{1}{4} G_2^\alpha \mathbb{H} + \frac{1}{2} \varepsilon^{\alpha\gamma} \mathbb{C}^\dagger Q_\gamma^1 \right) \otimes 1 \\
&\quad + 1 \otimes \left(\hat{G}_2^\alpha - \frac{1}{2} G_2^\alpha \mathbb{R}_2^2 + \frac{1}{2} \mathbb{R}_2^1 G_1^\alpha - \frac{1}{2} G_2^\gamma \mathbb{L}_\gamma^\alpha - \frac{1}{4} G_2^\alpha \mathbb{H} + \frac{1}{2} \varepsilon^{\alpha\gamma} \mathbb{C}^\dagger Q_\gamma^1 \right) \\
&\quad - G_2^\alpha \otimes \mathbb{R}_2^2 + \mathbb{R}_2^1 \otimes G_1^\alpha - G_2^\gamma \otimes \mathbb{L}_\gamma^\alpha - \frac{1}{2} G_2^\alpha \otimes \mathbb{H} + \varepsilon^{\alpha\gamma} \mathbb{C}^\dagger \otimes Q_\gamma^1, \\
\Delta \tilde{C} &= \left(\hat{C} + \frac{1}{2} \mathbb{C} \mathbb{H} \right) \otimes 1 + 1 \otimes \left(\hat{C} + \frac{1}{2} \mathbb{C} \mathbb{H} \right) + \mathbb{C} \otimes \mathbb{H}, \\
\Delta \tilde{C}^\dagger &= \left(\hat{C}^\dagger - \frac{1}{2} \mathbb{C}^\dagger \mathbb{H} \right) \otimes 1 + 1 \otimes \left(\hat{C}^\dagger - \frac{1}{2} \mathbb{C}^\dagger \mathbb{H} \right) - \mathbb{C}^\dagger \otimes \mathbb{H},
\end{aligned} \tag{B.1}$$

where one can observe the co-ideal property explicitly.

For the second, toy case, where $\mathfrak{g} = \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ and $\mathfrak{h} = \mathfrak{su}(1|2)$ is the subalgebra preserved on the boundary, the complete expressions of the co-products are

$$\begin{aligned}
\Delta \tilde{L}_3^4 &= \left(\hat{L}_3^4 - \frac{1}{2} \mathbb{L}_3^4 \mathbb{L}_4^4 + \frac{1}{2} \mathbb{L}_3^4 \mathbb{L}_3^3 + \frac{1}{2} G_c^4 Q_3^c \right) \otimes 1 \\
&\quad + 1 \otimes \left(\hat{L}_3^4 - \frac{1}{2} \mathbb{L}_3^4 \mathbb{L}_4^4 + \frac{1}{2} \mathbb{L}_3^4 \mathbb{L}_3^3 + \frac{1}{2} G_c^4 Q_3^c \right) \\
&\quad - \mathbb{L}_3^4 \otimes \mathbb{L}_4^4 + \mathbb{L}_3^4 \otimes \mathbb{L}_3^3 + G_c^4 \otimes Q_3^c, \\
\Delta \tilde{L}_4^3 &= \left(\hat{L}_4^3 - \frac{1}{2} \mathbb{L}_4^3 \mathbb{L}_3^3 + \frac{1}{2} \mathbb{L}_4^3 \mathbb{L}_4^4 + \frac{1}{2} Q_4^c G_c^3 \right) \otimes 1 \\
&\quad + 1 \otimes \left(\hat{L}_4^3 - \frac{1}{2} \mathbb{L}_4^3 \mathbb{L}_3^3 + \frac{1}{2} \mathbb{L}_4^3 \mathbb{L}_4^4 + \frac{1}{2} Q_4^c G_c^3 \right) \\
&\quad - \mathbb{L}_4^3 \otimes \mathbb{L}_3^3 + \mathbb{L}_4^3 \otimes \mathbb{L}_4^4 + Q_4^c \otimes G_c^3, \\
\Delta \tilde{Q}_4^a &= \left(\hat{Q}_4^a + \frac{1}{2} Q_4^c \mathbb{R}_c^a + \frac{1}{2} Q_4^a \mathbb{L}_4^4 - \frac{1}{2} \mathbb{L}_4^3 Q_3^a + \frac{1}{4} Q_4^a \mathbb{H} - \frac{1}{2} \varepsilon^{ad} \mathbb{C} G_d^3 \right) \otimes 1 \\
&\quad + 1 \otimes \left(\hat{Q}_4^a + \frac{1}{2} Q_4^c \mathbb{R}_c^a + \frac{1}{2} Q_4^a \mathbb{L}_4^4 - \frac{1}{2} \mathbb{L}_4^3 Q_3^a + \frac{1}{4} Q_4^a \mathbb{H} - \frac{1}{2} \varepsilon^{ad} \mathbb{C} G_d^3 \right) \\
&\quad + Q_4^c \otimes \mathbb{R}_c^a + Q_4^a \otimes \mathbb{L}_4^4 - \mathbb{L}_4^3 \otimes Q_3^a + \frac{1}{2} Q_4^a \otimes \mathbb{H} - \varepsilon^{ad} \mathbb{C} \otimes G_d^3, \\
\Delta \tilde{G}_a^4 &= \left(\hat{G}_a^4 - \frac{1}{2} G_c^4 \mathbb{R}_a^c - \frac{1}{2} G_a^4 \mathbb{L}_4^4 + \frac{1}{2} \mathbb{L}_3^4 G_a^3 - \frac{1}{4} G_a^\alpha \mathbb{H} + \frac{1}{2} \varepsilon_{ac} \mathbb{C}^\dagger Q_3^c \right) \otimes 1 \\
&\quad + 1 \otimes \left(\hat{G}_a^4 - \frac{1}{2} G_c^4 \mathbb{R}_a^c - \frac{1}{2} G_a^4 \mathbb{L}_4^4 + \frac{1}{2} \mathbb{L}_3^4 G_a^3 - \frac{1}{4} G_a^\alpha \mathbb{H} + \frac{1}{2} \varepsilon_{ac} \mathbb{C}^\dagger Q_3^c \right) \\
&\quad - G_c^4 \otimes \mathbb{R}_a^c - G_a^4 \otimes \mathbb{L}_4^4 + \mathbb{L}_3^4 \otimes G_a^3 - \frac{1}{2} G_a^\alpha \otimes \mathbb{H} + \varepsilon_{ac} \mathbb{C}^\dagger \otimes Q_3^c, \\
\Delta \tilde{C} &= \left(\hat{C} + \frac{1}{2} \mathbb{C} \mathbb{H} \right) \otimes 1 + 1 \otimes \left(\hat{C} + \frac{1}{2} \mathbb{C} \mathbb{H} \right) + \mathbb{C} \otimes \mathbb{H}, \\
\Delta \tilde{C}^\dagger &= \left(\hat{C}^\dagger - \frac{1}{2} \mathbb{C}^\dagger \mathbb{H} \right) \otimes 1 + 1 \otimes \left(\hat{C}^\dagger - \frac{1}{2} \mathbb{C}^\dagger \mathbb{H} \right) - \mathbb{C}^\dagger \otimes \mathbb{H}.
\end{aligned} \tag{B.2}$$

All generators in the right factors of the expressions above annihilate the boundary vacuum state $|0_B\rangle$; hence the reflection matrices do not implicitly depend on them. How-

ever, we do not expect these full co-products to be valid for boundaries *with* degrees of freedom, for the presence of the latter would change the boundary algebra structure. Further considerations are beyond the scope of the present work, but are the subject of current investigations.

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